1. The Online Encyclopedia of Integer Sequences (https://oeis.org) is a valuable computer resource for discovering and exploring patterns or hidden connections in integer sequences. For instance, sequence A000040 corresponds to the sequence of prime numbers, and sequence A000045 corresponds to terms of the Fibonacci sequence. In this problem, we will investigate a sequence that arises from partitioning the elements of a particular set.

Let \( p(n) \) denote the number of ways to partition the set \( S = \{1, 2, \ldots, 2n\} \) into \( n \) subsets each of size 2.

(a) Determine \( p(1) \) and \( p(2) \) directly.

**Solution:**

For \( n = 1 \), we see immediately that \( p(1) = 1 \), since there is one way to partition a 2-element set into one subset of size 2.

For \( n = 2 \), we must partition a 4-element set into two subsets each of size 2. Observe that once we determine the elements in one subset, the elements in the other subset are immediately determined. Thus, if we fix one element of the first subset, we only need to determine the number of choices for the other element of the first subset to obtain the answer. Fixing one element of the first subset gives us three choices for the second element of the first subset, so \( p(2) = 3 \).

(b) Develop a formula to calculate \( p(n) \) for all \( n \geq 2 \).

**Solution:**

To partition a \( 2n \)-element set into \( n \) subsets each of size 2, we begin by selecting the first subset of size 2. We can do this in \( \binom{2n}{2} \) ways. We then select a second subset of size 2. We can do this in \( \binom{2n-2}{2} = \binom{2(n-1)}{2} \) ways. Continuing in this manner, we find that there is a total of \( \binom{2n}{2} \binom{2(n-1)}{2} \cdots \binom{2(2)}{2} \binom{2(1)}{2} \) ways to construct \( n \) subsets each of size 2. Collectively, these subsets form a partition of the \( 2n \)-element set. Since there are \( n! \) ways to choose two elements from the \( 2n \)-element set at a time, and since each of these ways corresponds to the same partition of the set, we get that

\[
p(n) = \frac{\binom{2n}{2} \binom{2(n-1)}{2} \cdots \binom{2(2)}{2} \binom{2(1)}{2}}{n!}
= \frac{(2n)!}{2^n \cdot n!}
\]

(c) Visit https://oeis.org and enter a few terms of the sequence \( p(n) \) as given by your formula. Which sequence is generated by your formula? Give the name and number of the sequence. (You will need to enter four or five terms to narrow your search sufficiently.)

**Solution:**

The sequence generated by the formula for \( p(n) \) obtained in part (b) corresponds to A001147, the sequence of the double factorial of odd numbers: \( a(n) = (2n-1)!! \).

(The double factorial function \( n!! \) is a variant of the factorial function that takes the product of all numbers between 1 and \( n \) that have the same parity as \( n \).)

2. Analyze the following probabilistic argument. Is it correct? Explain why or why not.

A member of the tech support staff is updating equipment in the server room. The probability of successfully updating the equipment is \( 1/3 \) and each update attempt is independent. The staff member will make three attempts to update the equipment before quitting. Since the probability of a successful update is \( 1/3 = 0.33 \ldots \), the probability of a successful update after
three attempts is $3(1/3) = 0.99\ldots = 1$, so the staff member is certain to succeed in updating the equipment.

**Solution:** The argument is not correct.

Let $E_i$ denote the event “the staff member successfully updates the equipment on attempt $i$”. The error in the argument comes from assuming that $P[E_2] = P[E_3] = 1/3$. Indeed, this is not the case, since if the staff member succeeds in an earlier attempt, the later attempts are not performed.

It is true that $P[E_1] = 1/3$. However,

$$P[E_2] = P[\text{attempt 1 fails and attempt 2 succeeds}] = P[\text{attempt 1 fails}] \times P[\text{attempt 2 succeeds}] = 2/3 \times 1/3 = 2/9$$

and, similarly,

$$P[E_3] = P[\text{attempt 1 fails}] \times P[\text{attempt 2 fails}] \times P[\text{attempt 3 succeeds}] = 2/3 \times 2/3 \times 1/3 = 4/27.$$

Since each update attempt is independent, the probability of a successful update is

$$P[E_1 \cup E_2 \cup E_3] = P[E_1] + P[E_2] + P[E_3] = 1/3 + 2/9 + 4/27 = 19/27 = 0.703\ldots$$

[5 marks] 3. In this question, we will consider the following problem in computational complexity theory:

**Max-Sat**

*Given:* a Boolean formula in conjunctive normal form

Determine: an assignment of truth values that maximizes the number of true clauses in the formula.

The Max-Sat (maximum satisfiability) problem is similar to the Boolean satisfiability problem $Sat$, except instead of finding an assignment that makes every clause true, we want to find an assignment that maximizes the number of true clauses. Thus, Max-Sat is a generalization of $Sat$.

As an example, consider the formula $(x_0 \lor x_1) \land (x_0 \lor \neg x_1) \land (\neg x_0 \lor x_1) \land (\neg x_0 \lor \neg x_1)$. This formula is in conjunctive normal form; each clause contains only variables and $\lor$s, and all clauses are joined only by $\land$s. There is no way to satisfy all four clauses of this formula at once, but it is possible to satisfy three out of the four clauses.

(a) Suppose a formula in conjunctive normal form consists of $n$ variables, and we assign truth values to each variable randomly by flipping a fair coin. If we assign $T$ after flipping tails and $F$ after flipping heads, what is the probability of each possible assignment of truth values to the $n$ variables?

**Solution:** Each of the $n$ variables can take one of two possible truth values: $T$ or $F$. Each of these truth values has a probability of $1/2$ of being assigned to a given variable. Therefore, the probability of each possible assignment of truth values to $n$ variables is $1/2 \times \cdots \times 1/2 = 1/2^n$.

(b) Suppose a formula in conjunctive normal form is such that each clause is a disjunction of exactly two distinct variables or their negations; that is, each clause is of the form $(\omega \lor \eta)$, where the blank spaces contain variables. What is the probability that a given clause is true, if we assign truth values randomly by flipping a fair coin as in part (a)?

**Solution:** Given a clause of the form $(\omega \lor \eta)$, where the blank spaces contain variables, we have a total of $2^2 = 4$ possible assignments of truth values: two choices each for two variables. Out of these assignments, only one results in the clause being false. Therefore, the probability that the clause is true is $3/4$. 

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Since each update attempt is independent, the probability of a successful update is

$$P[E_1 \cup E_2 \cup E_3] = P[E_1] + P[E_2] + P[E_3] = 1/3 + 2/9 + 4/27 = 19/27 = 0.703\ldots$$

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(b) Suppose a formula in conjunctive normal form is such that each clause is a disjunction of exactly two distinct variables or their negations; that is, each clause is of the form $(\omega \lor \eta)$, where the blank spaces contain variables. What is the probability that a given clause is true, if we assign truth values randomly by flipping a fair coin as in part (a)?

**Solution:** Given a clause of the form $(\omega \lor \eta)$, where the blank spaces contain variables, we have a total of $2^2 = 4$ possible assignments of truth values: two choices each for two variables. Out of these assignments, only one results in the clause being false. Therefore, the probability that the clause is true is $3/4$. 

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(c) Suppose a formula in conjunctive normal form consists of $c$ clauses as defined in part (b). What is the expected number of true clauses in the formula, if we assign truth values randomly by flipping a fair coin as in part (a)?

**Solution:** Let $X$ denote the number of true clauses out of the $c$ total clauses in the formula. We have that $X = \sum_{i=1}^{c} X_i$, where $X_i = 1$ if the $i$th clause of the formula is true and $X_i = 0$ if the $i$th clause of the formula is false. The expected value of $X$ is therefore

$$E[X] = \sum_{i=1}^{c} E[X_i] = \sum_{i=1}^{c} P[X_i = 1] = \frac{3c}{4}.$$ 

[5 marks] 4. The “all-pegs” Tower of Hanoi puzzle is a variant of the classical puzzle where we must transfer $n$ disks from the first peg to the third peg, but without jumping directly from peg 1 to peg 3. In other words, our solution must use peg 2 whenever we move a disk from peg 1 to peg 3. All other aspects of the puzzle variant are identical to the original.

(a) Find a recurrence relation for $H'_n$, the number of moves needed to solve the “all-pegs” Tower of Hanoi puzzle with $n$ disks.

**Solution:** For $n = 0$, we have no disks to move, so $H'_0 = 0$. For $n = 1$, we have one disk to move. We must move this disk from peg 1 to peg 2, and then from peg 2 to peg 3, so $H'_1 = 2$.

Consider $n = k$ for some $k \geq 0$. Begin by moving the top $k - 1$ disks on peg 1 to peg 3. This step requires $H'_{k-1}$ moves. Next, move the last (and largest) disk from peg 1 to peg 2. Move the $k - 1$ disks from peg 3 back to peg 1, again in $H'_{k-1}$ moves. Then, move the last (and largest) disk from peg 2 to peg 3. Finally, move the $k - 1$ disks from peg 1 back to peg 3, again in $H'_{k-1}$ moves. Altogether, this procedure requires $H'_k = H'_{k-1} + 1 + H'_{k-1} + 1 + H'_{k-1} = 3H'_{k-1} + 2$ moves, so our recurrence relation is therefore

$$H'_n = 3H'_{n-1} + 2.$$ 

(b) Solve your recurrence relation from part (a) to obtain a formula for the number of moves needed to solve the “all-pegs” Tower of Hanoi puzzle with $n$ disks.

**Solution:** We use an iterative approach to solve the recurrence relation for $H'_n$.

$$H'_n = 3H'_{n-1} + 2$$

$$= 3 \left(3H'_n + 3(2) + 2 + 2\right) + 2 = 3^2H'_n + 3(2) + 2$$

$$= 3^2 \left(3H'_{n-2} + 3(2) + 2\right) + 3(2) + 2 = 3^3H'_n + 3^2(2) + 3(2) + 2$$

$$\vdots$$

$$= 3^{n-1}H'_1 + 3^{n-2}(2) + \cdots + 3^2(2) + 3(2) + 2$$

$$= 3^{n-1}(2) + 3^{n-2}(2) + \cdots + 3^2(2) + 3(2) + 2$$

$$= 2 \times \left(\sum_{i=1}^{n-1} 3^i\right) + 2$$

$$= 2 \times \left(\frac{1}{2}(3^n - 3)\right) + 2$$

$$= (3^n - 3) + 2$$

$$= 3^n - 1.$$